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A NEW CLASS OF PARTIALLY DEGENERATE LAGUERRE-BASED HERMITE-GENOCCHI POLYNOMIALS

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Abstract. In this paper, we introduce partially degenerate Laguerre-based Hermite-Genocchi and investigate their properties and identities. Furthermore, we introduce a generalized form of partially degenerate Laguerre-based Hermite-Genocchi and derive some interesting properties and identities. The results obtained are of general character and can be reduced to yield formulas and identities for relatively simple polynomials and numbers.

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1. Introduction

The generating function of the two variable Laguerre polynomials (2-VLP) $L_n(x,y)$ are defined by (see [2]):

$$\frac{1}{1-yt}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty}L_n(x,y)t^n;(\mid yt\mid<1). \tag{1.1}$$

These polynomials $L_n(x,y)$ are also given by the following generating function (see [3]):

$$\exp(yt)C_0(xt)=\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}, \tag{1.2}$$

where $C_0(x)$ is the 0^{th} order Tricomi function.

The n^{th} order Tricomi function is defined as (see [5])

$$C_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r!)}, \tag{1.3}$$

with the following generating function under condition for $t\neq 0$ and for all finite x.

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=0}^{\infty}C_n(x)t^n. \tag{1.4}$$

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Also, the Tricomi functions $C_n(x)$ are characterized by the following relation with the Bessel function $J_n(x)$ (see [5]):

$$C_n(x) = x^{\frac{n}{2}} J_n(2\sqrt{x}). \tag{1.5}$$

From equations (1.2) and (1.3), we obtain

$$L_n(x, y) = n! \sum_{s=0}^{n} \frac{(-1)^s \ x^s \ y^{n-s}}{(s!)^2 \ (n-s)!} = y^n L_n(x, y). \tag{1.6}$$

Thus, we have

$$L_n(x, y) = \frac{(-1)^n x^n}{n!}; \ L_n(0, y) = y^n; \ L_n(x, 1) = L_n(x), \tag{1.7}$$

where $L_n(x)$ are the ordinary Laguerre polynomials.

The 2-variable Kampé de Fériet generalization of the Hermite polynomials [1] and [4] reads

$$H_n(x, y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.8}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, \tag{1.9}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y = -1$ and x is replaced by $2x$.

In (2000), Dattoli et al. [4, p.241] introduced the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x, y, z)$ as follows:

$${}_LH_n(x, y, z) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{z^k L_{n-2k}(x, y)}{k!(n-2k)!}. \tag{1.10}$$

The generating function of the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x, y, z)$ as follows

$$\frac{1}{1-zt} \exp\left(\frac{-xt}{1-zt} + \frac{-yt^2}{1-zt^2}\right) = \sum_{n=0}^{\infty} {}_LH_n(x, y, z) t^n, \tag{1.11}$$

or, equivalently,

$$\exp(yt + zt^2) C_0(xt) = \sum_{n=0}^{\infty} {}_LH_n(x, y, z) \frac{t^n}{n!}. \tag{1.12}$$

It is clear from the above equations that

$${}_LH_n(x, y, -\frac{1}{2}) = {}_LH_n(x, y)$$

and

$${}_LH_n(x, 1, -1) = {}_LH_n(x),$$

where ${}_LH_n(x, y)$ and ${}_LH_n(x)$ denote the 2-variable Laguerre-Hermite polynomials (2VLHP) and Laguerre-Hermite polynomials, respectively (see [5]).

Kim-Kim [7] introduced the Daehee polynomials defined by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{(see [9, 10])}. \tag{1.13}$$

When $x = 0$, $D_n(0) = D_n$ are called the Daehee numbers.

For each $k \in \mathbb{N}_0$, the power of alternating sum $T_k(n)$ defined by (see [13]):

$$T_k(n) = \sum_{j=0}^{\infty}(-1)^j j^k. \tag{1.14}$$

The exponential generating function for $T_k(n)$ is

$$\sum_{k=0}^{\infty} T_k(n)\frac{t^k}{k!} = \frac{1-(-e^t)^{n+1}}{e^t+1}. \tag{1.15}$$

In (2017), Khan et al. [13] introduced a new class of partially degenerate Hermite-based Genocchi polynomials as follows:

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{1.16}$$

When $x = y = 0$, ${}_HG_{n,\lambda}(0,0) = G_{n,\lambda}$ are called the partially degenerate Genocchi numbers.

The idea of degenerate numbers and polynomials found existence with the study related to Bernoulli and Euler numbers and polynomials. Lately, many researchers have begun to study the degenerate versions of the classical and special polynomials (see [6, 8, 11, 12, 17, 18, 20, 22-25], for a systematic work). Influenced by their works, we introduce partially degenerate Laguerre-based Hermite-Genocchi polynomials and also a new generalization of partially degenerate Laguerre-based Hermite-Genocchi polynomials and then give some of their applications. We also derive some implicit summation formula and general symmetry identities.

2. Partially degenerate Laguerre-based Hermite-Genocchi polynomials

In this section, we introduce partially degenerate Laguerre-based Hermite-Genocchi polynomials and its properties. We begin with the following definition as follows.

We assume that $\lambda, t \in \mathbb{C}$ with $\mid \lambda t \mid \leq 1$ and $\lambda t \neq -1$. Then we consider partially degenerate Laguerre-based Hermite-Genocchi polynomials as follows:

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{yt+zt^2}C_0(xt) = \sum_{n=0}^{\infty} {}_{LH}G_{n,\lambda}(x,y,z)\frac{t^n}{n!}. \tag{2.1}$$

In the case, when $x = y = z = 0$, ${}_LHG_{n,\lambda}(0,0,0) = {}_LHG_{n,\lambda}$ are called the partially degenerate Laguerre-based Hermite-Genocchi numbers.

Theorem 2.1. For $n \in \mathbb{N}_0$ and $|\lambda t| < 1$, we have

$$ {}_LHG_{n,\lambda}(x,y,z) = \sum_{m=0}^{n} \frac{m!}{m+1} (-\lambda)^m \left(\begin{array}{c} n \\ m \end{array} \right) {}_LHG_{n-m,\lambda}(x,y,z). \tag{2.2} $$

Proof. It follows from (2.1) that

$$ \sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{yt+zt^2} C_0(xt). \tag{2.3} $$

Multiplying and dividing by t on the right hand side of (2.3), we have

$$ \sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!} = \left[\frac{\log(1+\lambda t)^{1/\lambda}}{t}\right] \left[\frac{2t}{e^t+1} e^{yt+zt^2} C_0(xt)\right] \tag{2.4} $$

$$ = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m+1}(\lambda t)^m\right) \left(\sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!}\right). \tag{2.5} $$

$$ \sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \frac{(-\lambda)^m}{m+1} m! {}_LHG_{n,\lambda}(x,y,z)\right) \frac{t^n}{n!}. \tag{2.6} $$

Comparing the coefficient of $\frac{t^n}{n!}$ in (2.6), we obtain at the desired result. □

Theorem 2.2. For $n \in \mathbb{N}_0$ and $|\lambda t| < 1$, we have

$$ {}_LHG_{n,\lambda}(x,y,z) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \lambda^m D_m {}_LHG_{n-m,\lambda}(x,y,z). \tag{2.7} $$

Proof. Consider (2.1), we have

$$ \sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!} = \frac{1}{t}\frac{2t\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{yt+zt^2} C_0(xt) $$

$$ = \left[\frac{\log(1+\lambda t)}{\lambda\ t}\right] \left[\frac{2t}{e^t+1} e^{yt+zt^2} C_0(xt)\right] $$

$$ = \left(\sum_{m=0}^{\infty} D_m \frac{(\lambda t)^m}{m!}\right) \left(\sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!}\right) $$

$$ = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \lambda^m D_m {}_LHG_{n-m,\lambda}(x,y,z)\right) \frac{t^n}{n!}. \tag{2.8} $$

Comparing the coefficients of t^n, we get (2.7). □

Theorem 2.3. For $n \in \mathbb{N}_0$ and $|\lambda t| < 1$, we have

$$ {}_LHG_{n,\lambda}(x,y,z) = n \sum_{m=0}^{n-1} \left(\begin{array}{c} n-1 \\ m \end{array} \right) \lambda^m {}_LHE_{n-m-1}(x,y,z) D_m, \tag{2.9} $$

where, ${}_LHE_{n,\lambda}(x,y,z)$ is Laguerre-based Hermite-Euler polynomials (see [2-4]).

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_LH G_{n,\lambda}(x,y,z)\frac{t^n}{n!} = t\left(\frac{\log(1+\lambda t)}{\lambda t}\right)\left(\frac{2}{e^t+1}e^{yt+zt^2}C_0(xt)\right)$$
$$= t\left(\sum_{m=0}^{\infty} D_m\frac{(\lambda t)^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_LH E_n(x,y,z)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\lambda^m D_m {}_LH E_{n-m}(x,y,z)\right)\frac{t^{n+1}}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-1} n\binom{n-1}{m}\lambda^m D_m {}_LH E_{n-m-1}(x,y,z)\right)\frac{t^n}{n!}. \tag{2.10}$$

Thus, by comparing the coefficients of t^n on both sides of the above equation, yields our required result. □

Theorem 2.4. For $n \in \mathbb{N}_0$ and $\mid \lambda t \mid < 1$, we have

$${}_LH G_{n,\lambda}(x,y+1,z) = \sum_{m=0}^{n}\binom{n}{m} {}_LH G_{n-m,\lambda}(x,y,z). \tag{2.11}$$

Proof. From (2.1), we see that

$$\sum_{n=0}^{\infty}\left[{}_LH G_{n,\lambda}(x,y+1,z) - {}_LH G_{n,\lambda}(x,y,z)\right]\frac{t^n}{n!}$$
$$= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{(y+1)t+zt^2}C_0(xt) - \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{yt+zt^2}C_0(xt)$$
$$= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{1+e^t}e^{yt+zt^2}(e^t-1)C_0(xt)$$
$$= \sum_{n=0}^{\infty} {}_LH G_{n,\lambda}(x,y,z)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_LH G_{n,\lambda}(x,y,z)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} {}_LH G_{n,\lambda}(x,y,z) - {}_LH G_{n,\lambda}(x,y,z)\right)\frac{t^n}{n!}. \tag{2.12}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get our desired result (2.11). □

Theorem 2.5. For $n \in \mathbb{N}_0$ and $\mid \lambda t \mid < 1$, we have

$${}_LH G_{n,\lambda}(x,y,z) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m} D_m\lambda^m {}_LH_{k-m}(x,y,z)G_{n-m}. \tag{2.13}$$

Proof. Using equations (1.10), (1.13) and (2.1), we have

$$\sum_{n=0}^{\infty} {}_LH G_{n,\lambda}(x,y,z)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{yt+zt^2}C_0(xt)$$
$$= \left(\frac{2t}{e^t+1}\right)\left(\frac{\log(1+\lambda t)}{\lambda t}\right)e^{yt+zt^2}C_0(xt)$$

$$= \left(\sum_{n=0}^{\infty} G_n \frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} D_m \frac{(\lambda t)^m}{m!}\right)\left(\sum_{k=0}^{\infty} {}_LH_k(x,y,z)\frac{t^k}{k!}\right)$$

$$= \left(\sum_{n=0}^{\infty} G_n \frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m} D_m \lambda^m {}_LH_{k-m}(x,y,z)\frac{t^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m} D_m \lambda^m {}_LH_{k-m}(x,y,z) G_{n-m}\right)\frac{t^n}{n!}. \tag{2.14}$$

Comparing the coefficients of t^n, we get (2.13) □

Theorem 2.6. For $n \in \mathbb{N}_0$, we have

$$_{L}{}_HG_{n,\lambda}(x,y,z) = d^{n-1}\sum_{a=0}^{d-1} {}_LHG_{n,\frac{\lambda}{d}}\left(\frac{a+y}{d}, z, x\right). \tag{2.15}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,y,z)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{yt+zt^2} C_0(xt)$$

$$= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1} e^{zt^2} C_0(xt) \sum_{a=0}^{d-1} e^{(a+y)t}$$

$$= \sum_{n=0}^{\infty}\left(d^n \sum_{a=0}^{d-1} {}_LHG_{n,\frac{\lambda}{d}}\left(\frac{a+y}{d}, z, x\right)\right)\frac{t^n}{n!}. \tag{2.16}$$

Equating the coefficients of $\frac{t^n}{n!}$ in above equation, we get the required result. □

Remark 2.1. On setting $x = 0$, Theorem 2.6 reduces to the known results of Khan et al. [13].

3. Summation formulae

In this section, we give some implicit summation formula of partially degenerate Laguerre-based Hermite-Genocchi polynomials by making use of summation technique methods.

Theorem 3.1. The following implicit summation formula for partially degenerate Laguerre-based Hermite-Genocchi polynomials holds true:

$$_{L}{}_HG_{k+l,\lambda}(x,s,z) = \sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}(s-y)^{n+p} {}_LHG_{k+l-p-n,\lambda}(x,y,z). \tag{3.1}$$

Proof. We first replace t by $t + u$ and again writing the generating function (2.1), we get

$$\frac{2\log(1+\lambda(t+u))^{\frac{1}{\lambda}}}{e^{(t+u)}+1} e^{z(t+u)^2} C_0(x(t+u)) = e^{-y(t+u)} \sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,y,z)\frac{t^k u^l}{k!l!}.$$

Replacing y by s in the above equation and equating the resulting expression with the original equation, we get

$$e^{(s-y)(t+u)}\sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,s,z)\frac{t^k}{k!}\frac{u^l}{l!}$$

$$\sum_{N=0}^{\infty}\frac{[(s-y)(t+u)]^N}{N!}\sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,s,z)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.2}$$

$$\sum_{n,p=0}^{\infty}\frac{(s-y)^{n+p}t^n u^p}{n!p!}\sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,s,z)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.3}$$

Now replacing k by $k-n$, l by $l-p$ in above equation, we get

$$\sum_{n,p=0}^{\infty}\sum_{k,l=0}^{n,p}\frac{(s-y)^{n+p}}{n!p!} {}_LHG_{k+l-n-p,\lambda}(x,y,z)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!}$$

$$= \sum_{k,l=0}^{\infty} {}_LHG_{k+l,\lambda}(x,s,z)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.4}$$

Finally, on equating the coefficients of the like powers of t^k and u^l in the above equation, we arrive at the desired result. □

Corollary 3.1. For $l = 0$, the following implicit summation formulae for partially degenerate Laguerre-based Hermite-Genocchi polynomials ${}_LHG_{n,\lambda}(x,y,z)$ holds true:

$${}_LHG_{k,\lambda}(x,s,z) = \sum_{j=0}^{k}\binom{k}{j}(s-y)^j {}_LHG_{k-n,\lambda}(x,y,z). \tag{3.5}$$

Theorem 3.2. The following implicit summation formula for partially degenerate Laguerre-based Hermite-Genocchi polynomials ${}_LHG_{n,\lambda}(x,y,z)$ holds true:

$${}_LHG_{n,\lambda}(x,y+u,z+v) = \sum_{m=0}^{n}\binom{n}{m} {}_LHG_{n-m,\lambda}(u,v)H_m(y,z). \tag{3.6}$$

Proof. Replacing y by $(y+u)$ and z by $(z+v)$ in (2.1), we have

$$\sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(y+u,z+v,x)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{(y+u)t+(z+v)t^2}C_0(xt)$$

$$= \left(\sum_{n=0}^{\infty} {}_LHG_{n,\lambda}(x,u,v)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(y,z)\frac{t^m}{m!}\right). \tag{3.7}$$

Finally, replacing n by $n-m$ and after comparing the coefficients of $\frac{t^n}{n!}$, we get the desired result. □

Theorem 3.3. The following implicit summation formula for partially degenerate Laguerre-based Hermite-Genocchi polynomials holds true:

$$ {}_{L}{}_{H}G_{n,\lambda}(x,y,z) \;=\; n!\sum_{k=0}^{\frac{n}{2}} {}_{L}G_{n-2k,\lambda}(x,z)\frac{y^{k}}{(n-2k)!k!}. \tag{3.8}$$

Proof. From(2.1), we get

$$\sum_{n=0}^{\infty} {}_{L}{}_{H}G_{n,\lambda}(x,z,y)\frac{t^{n}}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{t}+1}e^{zt+yt^{2}}C_{0}(xt)$$

$$= \sum_{n=0}^{\infty} {}_{L}G_{n,\lambda}(z,x)\frac{t^{n}}{n!}\sum_{k=0}^{\infty} y^{k}\frac{t^{2k}}{k!}. \tag{3.9}$$

Now replacing n by $n-2k$ in the above equation and comparing the coefficients of t^{n}, we get (3.8). □

Theorem 3.4. The following implicit summation formulae for partially degenerate Laguerre-based Hermite-Genocchi polynomials ${}_{L}{}_{H}G_{n,\lambda}(x,y,z)$ holds true:

$$ {}_{L}{}_{H}G_{n,\lambda}(x,y,z) \;= \sum_{m=0}^{n}\binom{n}{m} {}_{L}G_{n-m,\lambda}(y-s)H_{m}(s,z). \tag{3.10}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_{L}{}_{H}G_{n,\lambda}(x,y,z)\frac{t^{n}}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{1+e^{t}}e^{(y-s)t}e^{st+zt^{2}}C_{0}(xt)$$

$$= \sum_{n=0}^{\infty} {}_{L}G_{n,\lambda}(y-s,x)\frac{t^{n}}{n!}\sum_{m=0}^{\infty} H_{m}(s,z)\frac{t^{m}}{m!}$$

$$\sum_{n=0}^{\infty} {}_{L}{}_{H}G_{n,\lambda}(x,y,z)\frac{t^{n}}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_{L}G_{n-m,\lambda}(y-s,x)H_{m}(s,z)\frac{t^{n}}{n!}. \tag{3.11}$$

Finally, on comparing the coefficients of like powers of t, we get the desired result. □

Theorem 3.5. The following implicit summation formulae for partially degenerate Laguerre-based Hermite-Genocchi polynomials ${}_{L}{}_{H}G_{n,\lambda}(x,y,z)$ holds true:

$$ {}_{L}{}_{H}G_{n,\lambda}(y+1,z,x) \;=\; \sum_{m=0}^{n}\binom{n}{m} {}_{L}{}_{H}G_{n-m,\lambda}(y,z,x). \tag{3.12}$$

Proof. Replacing y by $y+1$ in (2.1), we get

$$\sum_{n=0}^{\infty} {}_{L}{}_{H}G_{n,\lambda}(y+1,z,x)\frac{t^{n}}{n!} \;=\; \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{1+e^{t}}e^{(y+1)t+zt^{2}}C_{0}(xt)$$

$$= \left(\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{1+e^{t}}e^{yt+zt^{2}}\right)C_{0}(xt)e^{t}$$

$$= \sum_{n=0}^{\infty} {}_{L}HG_{n,\lambda}(x,y,z)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!}. \tag{3.13}$$

Replacing n by $n-m$ in the above equation and comparing the coefficients of t, we get the desired result.

□

Theorem 3.6 The following implicit summation formulae for partially degenerate Laguerre-based Hermite-Genocchi polynomials ${}_{L}HG_{n,\lambda}(x,y,z)$ holds true:

$${}_{L}HG_{n,\lambda}(y+1,z,x)+{}_{L}HG_{n,\lambda}(y,z,x) = 2n\sum_{m=0}^{n-1}\binom{n-1}{m}\frac{(-\lambda)^m m!}{m+1}{}_{L}H_{n-1-m}(x,y,z). \tag{3.14}$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty}\left[{}_{L}HG_{n,\lambda}(y+1,z,x)+{}_{L}HG_{n,\lambda}(y,z,x)\right]\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{(y+1)t+zt^2}C_0(xt)$$

$$+\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{yt+zt^2}C_0(xt)$$

$$= 2\log(1+\lambda t)^{\frac{1}{\lambda}}e^{yt+zt^2}C_0(xt)$$

$$\sum_{n=0}^{\infty}\left[{}_{L}HG_{n,\lambda}(y+1,z,x)+{}_{L}HG_{n,\lambda}(y,z,x)\right]\frac{t^n}{n!} = 2t\left(\frac{\log(1+\lambda t)}{\lambda t}\right)e^{yt+zt^2}C_0(xt)$$

$$= 2t\left(\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda t)^m}{m+1}\right)\left(\sum_{n=0}^{\infty}{}_{L}H_n(y,z,x)\frac{t^n}{n!}\right)$$

$$\sum_{n=0}^{\infty}\left[{}_{L}HG_{n,\lambda}(y+1,z,x)+{}_{L}HG_{n,\lambda}(y,z,x)\right]\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{(-\lambda)^m m!}{m+1}$$

$$\times{}_{L}H_{n-m}(x,y,z)\frac{t^{n+1}}{n!}. \tag{3.15}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get the required result. □

4. Symmetry identities

Symmetry identities involving various polynomials have been discussed (e.g., [14-16, 19, 21]). As in above-cited work, here, in view of the generating functions (1.16) and (2.1), we obtain symmetry identities for the partially degenerate Laguerre-based Hermite-Genocchi polynomials ${}_{L}HG_{n,\lambda}(x,y,z)$.

Theorem 4.1. For each pair of integers a and b with $n \geq 0$, the following symmetry identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}b^m a^{n-m}{}_{L}HG_{n-k,\lambda}(by,b^2z,bx)\ {}_{L}HG_{m,\lambda}(ay,a^2z,ax)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_{LH}G_{n-k,\lambda}(ay, a^2z, ax)\ {}_{LH}G_{m,\lambda}(by, b^2z, bx). \quad (4.1)$$

Proof. We first consider

$$f(t) = \frac{(2\log(1+\lambda at)^{\frac{a}{\lambda}})(2\log(1+\lambda bt)^{\frac{b}{\lambda}})}{(e^{at}+1)(e^{bt}+1)} e^{abyt+a^2b^2zt^2} C_0(abxt), \quad (4.2)$$

where $f(t)$ is symmetric in a and b and can be expressed into series in two ways.

$$f(t) = \sum_{n=0}^{\infty} {}_{LH}G_{n,\lambda}(by, b^2z, bx) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_{LH}G_{m,\lambda}(ay, a^2z, ax) \frac{(bt)^m}{m!}$$

$$f(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_{LH}G_{n-k,\lambda}(by, b^2z, bx) {}_{LH}G_{m,\lambda}(ay, a^2z, ax) \right) \frac{t^n}{n!}. \quad (4.3)$$

On the other hand,

$$f(t) = \sum_{n=0}^{\infty} {}_{LH}G_{n,\lambda}(ay, a^2z, ax) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_{LH}G_{m,\lambda}(by, b^2z, bx) \frac{(at)^m}{m!}$$

$$f(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_{LH}G_{n-k,\lambda}(ay, a^2z, ax) {}_{LH}G_{m,\lambda}(by, b^2z, bx) \right) \frac{t^n}{n!}. \quad (4.4)$$

Therefore by (4.3) and (4.4), we get (4.1). □

Theorem 4.2. For each pair of integers a and b with $n \geq 1$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_{LH}G_{n-k,\lambda} \left(by + \frac{b}{a} i + j, b^2z, bx \right) {}_{L}G_{k,\lambda}(az, ax)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} {}_{LH}G_{n-k,\lambda} \left(ay + \frac{a}{b} i + j, a^2z, ax \right) {}_{L}G_{k,\lambda}(bz, bx). \quad (4.5)$$

Proof. Let us consider,

$$h(t) = \frac{(2\log(1+\lambda t)^{\frac{a}{\lambda}})(2\log(1+\lambda t)^{\frac{b}{\lambda}})(e^{abt}+1)^2}{(e^{at}+1)^2(e^{bt}+1)^2} e^{ab(y+u)t+a^2b^2zt^2} [C_0(abxt)]^2. \quad (4.6)$$

$$= \frac{2\log(1+\lambda t)^{\frac{a}{\lambda}}}{e^{at}+1} e^{abyt+a^2b^2zt^2} C_0(abxt) \left(\frac{e^{abt}+1}{e^{bt}+1} \right)$$

$$\times \frac{2\log(1+\lambda t)^{\frac{b}{\lambda}}}{e^{bt}+1} e^{abut} C_0(abxt) \left(\frac{e^{abt}+1}{e^{at}+1} \right)$$

$$= \frac{2\log(1+\lambda t)^{\frac{a}{\lambda}}}{e^{at}+1} e^{abyt+a^2b^2zt^2} C_0(abxt) \left(\sum_{i=0}^{a-1} (-1)^i e^{bti} \right)$$

$$\times\frac{2\log(1+\lambda t)^{\frac{b}{\lambda}}}{e^{bt}+1}e^{abut}C_0(abxt)\left(\sum_{j=0}^{b-1}(-1)^j e^{atj}\right)$$

$$=\left(\sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^{i+j}{}_{LH}G_{n,\lambda}\left(by+\frac{b}{a}i+j,b^2z,bx\right)\frac{(at)^n}{n!}\right)$$

$$\times\sum_{k=0}^{\infty}{}_{L}G_{k,\lambda}(au,ax)\frac{(bt)^k}{(k)!}$$

$$h(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^{i+j}\right.$$

$$\left.\times\ {}_{LH}G_{n-k,\lambda}\left(by+\frac{b}{a}i+j,b^2z,bx\right){}_{L}G_{k,\lambda}(au,ax)\right)\frac{t^n}{n!}. \tag{4.7}$$

On the other hand,

$$h(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^{i+j}\right.$$

$$\left.\times\ {}_{LH}G_{n-k,\lambda}\left(ay+\frac{a}{b}i+j,a^2z,ax\right){}_{L}G_{k,\lambda}(bu,bx)\right)\frac{t^n}{n!}. \tag{4.8}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (4.7) and (4.8), we arrive at the desired result. □

Theorem 4.3. For each pair of integers a and b and all integers $n \geq 0$, the following symmetry identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k{}_{LH}G_{n-k,\lambda}\left(by,b^2u,bx\right)\sum_{i=0}^{k}\binom{k}{i}T_i(a-1)G_{k-i,\lambda}(az)$$

$$=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}{}_{LH}G_{n-k,\lambda}\left(ay,a^2u,ax\right)\sum_{i=0}^{k}\binom{k}{i}T_i(b-1)G_{k-i,\lambda}(bz), \tag{4.9}$$

where the sum of alternative integer powers $T_k(n)$ is already given by (1.15).

Proof. Suppose

$$p(t)=\frac{(2\log(1+\lambda t)^{\frac{a}{\lambda}})(2\log(1+\lambda t)^{\frac{b}{\lambda}})(1-(-e^{bt})^a)e^{ab(y+z)t+a^2b^2ut^2}C_0(abxt)}{(e^{at}+1)(e^{bt}+1)^2} \tag{4.10}$$

$$p(t)=\left(\sum_{n=0}^{\infty}{}_{LH}G_{n,\lambda}(by,b^2u,bx)\frac{(at)^n}{n!}\right)\left(\sum_{i=0}^{\infty}T_i(a-1)\frac{(bt)^i}{i!}\right)\left(\sum_{k=0}^{\infty}G_{k,\lambda}(az)\frac{(bt)^k}{k!}\right). \tag{4.11}$$

Using a similar method in (4.10), we get

$$g=\left(\sum_{n=0}^{\infty}{}_{LH}G_{n,\lambda}(ay,a^2u,ax)\frac{(at)^n}{n!}\right)\left(\sum_{i=0}^{\infty}T_i(b-1)\frac{(at)^i}{i!}\right)\left(\sum_{k=0}^{\infty}G_{k,\lambda}(bz)\frac{(at)^k}{k!}\right). \tag{4.12}$$

After an appropriate change of summation index and comparison of the coefficients of $\frac{t^n}{n!}$ in (4.11) and (4.12), we get our required result. □

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